

# A combinatorial proof of Guo's multi-generalization of Munarini's identity

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**Abstract.** We give a combinatorial proof of Guo's multi-generalization of Munarini's identity, answering a question of Guo.

*Keywords:* multinomial coefficient, Munarini's identity, involutive proof.

## 1 Introduction

Simons [7] proved a binomial coefficient identity which is equivalent to

$$\sum_{k=0}^n \binom{n}{k} \binom{n+k}{k} (-1)^{n-k} (1+x)^k = \sum_{k=0}^n \binom{n}{k} \binom{n+k}{k} x^k. \quad (1.1)$$

Several different proofs of (1.1) were given by [1, 5, 8]. Using Cauchy's integral formula as in [5], Munarini [4] obtained the following generalization

$$\sum_{k=0}^n \binom{\beta - \alpha + n}{n-k} \binom{\beta + k}{k} (-1)^{n-k} (x+y)^k y^{n-k} = \sum_{k=0}^n \binom{\alpha}{n-k} \binom{\beta + k}{k} x^k y^{n-k}, \quad (1.2)$$

where  $\alpha, \beta, x$  and  $y$  are indeterminates. It is clear that (1.2) reduces to (1.1) when  $\alpha = \beta = n$  and  $y = 1$ . Shattuck [6] and Chen and Pang [2] provided two interesting combinatorial proofs of (1.2).

Recently, Guo [3] obtained the following multinomial coefficient generalization of (1.2) as follows:

$$\sum_{\mathbf{k}=0}^{\mathbf{n}} (-1)^{|\mathbf{n}|-|\mathbf{k}|} \binom{\beta - \alpha + |\mathbf{n}|}{\mathbf{n} - \mathbf{k}} \binom{\beta + |\mathbf{k}|}{\mathbf{k}} (\mathbf{x} + \mathbf{y})^{\mathbf{k}} \mathbf{y}^{\mathbf{n}-\mathbf{k}} = \sum_{\mathbf{k}=0}^{\mathbf{n}} \binom{\alpha}{\mathbf{n} - \mathbf{k}} \binom{\beta + |\mathbf{k}|}{\mathbf{k}} \mathbf{x}^{\mathbf{k}} \mathbf{y}^{\mathbf{n}-\mathbf{k}}, \quad (1.3)$$

where  $\mathbf{n} = (n_1, \dots, n_m) \in \mathbb{N}^m$ ,  $|\mathbf{n}| = n_1 + \dots + n_m$ ,  $\mathbf{x}, \mathbf{y} \in \mathbb{C}^m$ , the *multinomial coefficient*  $\binom{x}{\mathbf{n}}$  is defined by

$$\binom{x}{\mathbf{n}} = \begin{cases} \frac{x(x-1) \cdots (x-|\mathbf{n}|+1)}{n_1! \cdots n_m!}, & \text{if } \mathbf{n} \in \mathbb{N}^m, \\ 0, & \text{otherwise,} \end{cases}$$

and  $\mathbf{b}^{\mathbf{a}} = b_1^{a_1} \cdots b_m^{a_m}$  for  $\mathbf{a} = (a_1, \dots, a_m) \in \mathbb{N}^m$  and  $\mathbf{b} = (b_1, \dots, b_m) \in \mathbb{C}^m$ .

In this paper we shall give an involutive proof of (1.3), answering a question of Guo [3]. Our proof is motivated by Shattuck's work [6].

## 2 The Involution Proof

Note that both sides of (1.3) are polynomials in  $\alpha, \beta, x_1, \dots, x_m$  and  $y_1, \dots, y_m$ . We may only consider the case of positive integers with  $\beta \geq \alpha$ . We first understand the unsigned quantity in the sum of the left-hand side of (1.3). Let  $\Gamma = \{a, b_1, \dots, b_m\}$  be an alphabet. We construct the weighted words  $w = w_1 \cdots w_{\beta+|\mathbf{n}|}$  on  $\Gamma$  as follows:

- i) Choose  $\mathbf{k} = (k_1, \dots, k_m) \in \mathbb{N}^m$  with  $0 \leq k_i \leq n_i$ ;
- ii) Let a subword of  $w_1 \cdots w_{\beta-\alpha+|\mathbf{n}|}$  be a permutation of  $\{b_1^{n_1-k_1}, \dots, b_m^{n_m-k_m}\}$ , with each  $b_i$  weighted  $y_i$  and also circled;
- iii) Let all the other  $w_i$ 's be a permutation of  $\{a^\beta, b_1^{k_1}, \dots, b_m^{k_m}\}$ , with each  $b_i$  weighted  $x_i$  or  $y_i$  and each  $a$  weighted 1.

We call such a weighted words  $w$  a *configuration*, and define its weight as the product of the weights of all the  $w_i$ 's. It is not hard to see that the sum of the weights of the configurations defined above is equal to  $\binom{\beta-\alpha+|\mathbf{n}|}{\mathbf{n}-\mathbf{k}} \binom{\beta+|\mathbf{k}|}{\mathbf{k}} (\mathbf{x} + \mathbf{y})^{\mathbf{k}} \mathbf{y}^{\mathbf{n}-\mathbf{k}}$  for any  $\mathbf{k} \in \mathbb{N}^m$ .

Let  $S^+$  and  $S^-$  denote the sets of configurations with an even or odd number of circled positions, respectively. We pair members  $w$  of  $S^+$  and  $S^-$  by identifying the first  $w_j$  with weight  $y_i$  ( $1 \leq i \leq m$ ) in  $w_1 \cdots w_{\beta-\alpha+|\mathbf{n}|}$ , either circling it or uncircling it. Here we give an example for  $\beta = 4, \alpha = 2$  and  $\mathbf{n} = (2, 2)$ :

$$\begin{array}{cccccccccccccccc} a & b_2 & a & b_1 & \textcircled{b_2} & a & b_1 & a & \iff & a & b_2 & a & \textcircled{b_1} & \textcircled{b_2} & a & b_1 & a \\ x_2 & & & y_1 & y_2 & & x_1 & & & x_2 & & & y_1 & y_2 & & x_1 & \end{array}$$

This pairs all configurations except those where every position weighted  $y_i$  is in the right  $\alpha$  positions. These configurations  $w$  must belong to  $S^+$  and the sum of their weights is equal to the right-hand side of (1.3). This is because if the subwords with elements weighted  $y_i$  ( $1 \leq i \leq m$ ) in  $w_{\beta-\alpha+|\mathbf{n}|+1} \cdots w_{\beta+|\mathbf{n}|}$  is a permutation of  $\{b_1^{n_1-k_1}, \dots, b_m^{n_m-k_m}\}$ , then there are  $\binom{\beta+|\mathbf{k}|}{\mathbf{k}}$  possible ways to choose the remaining subwords of  $w$ , where each  $b_i$  is weighted  $x_i$ . This proves (1.3).

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